Berkeley Math Circle

GRAPH THEORY in OLYMPIAD PROBLEM SOLVING[1]

by Zvezdelina Stankova

Mills College and University of California at Berleley

September 26, 2006

1. Problems from *Tournament of the Towns, 1984-1989, 1989-1993*

edited by P.J. Taylor, publ. by Australian Mathematics Trust

(1) *(Spring'86, Junior Questions, Problem 5.)* 20 football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

(2) *(Autumn'92, Junior Questions, Level I, Problem 1.)* There are 101 chess players who participated in several tournaments. There was no tournament in which all of them participated. Each pair of these 101 players met exactly once during these tournaments. Prove that one of them participated in no less than 11 tournaments. (Assume that each pair of participants in each tournament plays each other once in that tournament.)

(3) *(Autumn'91, Junior Questions, Level II, Problem 1.)* 32 knights live in a kingdom. Some of them are servants of others. A servant may have only one master and any master is more wealthy than any of his servants. A knight having not less than 4 servants is called a baron. What is the maximum number of barons? (The kingdom is ruled by the law: "My servant's servant is not my servant.")

(4) *(Spring'93, Junior Questions, Level II, Problem 4.)* There are 25 students in Peter's class (not counting him). Peter has observed that all 25 have different numbers of friends in this class. How many friends does Peter have in this class? (Give all possible answers.)

(5) *(Spring'91, Senior Questions, Level II, Problem 5.)* There are 16 cities in a kingdom. The king wants to have a system of roads constructed so that one can go along those roads from any city to any other one without going through more than one intermediate city and so that no more than 5 roads go out of any city.

   (a) Prove that this is possible.
   (b) Prove that if we replace 5 by 4 in the problem, the king's desire will become unrealizable.

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[1] *Note:* The problems in this handout vary from medium hard to very difficult! This is **not** a regular school homework assignment, but rather a collection of olympiad problems, each of which may take hours or days to be solved. To the Beginners: just read the problems, try to understand what they are asking and draw pictures or diagrams to imagine better what is going on. Some of the terminology will be unfamiliar, so check out the chapters Graphs-1 and Graphs-2 in "Mathematical Circles" by Fomin, Genkin, Itenberg, published by AMS (this is one of the books that you can buy at discount at the Berkeley Math Circle). The Veterans in the Circle should also attempt to read the discussions and theorems in these chapters. In any case: do **not** be discouraged if you can't solve many, or indeed any, of the problems here. Treat this handout as a book written in an unknown language, which you will be learning throughout a series of exercises over a substantial period of time. The handout is intended to give you an idea of the type of problems that you will be working on at the Circle with the guidance of our instructors. We discourage you from peeking at the solutions to problems. But if, after days of hard work, you are absolutely dying to see how to solve these problems, go to

- *the Tournament of the Towns*, look further in this handout;
- the BAMO problems: look at the Berkeley Math Circle webpage *http://mathcircle.berkeley.edu*;
- the problems from *Mathematical Circles.*

2. Problems from the *Bay Area Mathematical Olympiad (BAMO)*

(6) *(BAMO'04. Problem 3.)* NASA has proposed populating Mars with 2,004 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing N tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of N that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

(7) *(BAMO'05. Problem 4.)* There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

(8) *(BAMO'06. Problem 1.)* All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:
  (a) Every student must move to a new chair.
  (b) A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules above allow two students in adjacent chairs to exchange places.)
Show that this procedure can be done if n is even, and cannot be done if n is odd.

3. Problems from *Mathematical Circles*: For the Die-Hards!

by Fomin, Genkin and Itenberg, American Mathematical Society, 1996

(9) (Problem 16*, p.142) Some of the 100 towns in a country are connected by airlines. It is known that one can reach every town from any other (perhaps with several intermediate stops). Prove that you can fly around the country and visit all the towns making no more than: (a) 198 flights; (b) 196 flights.

(10) (Problem 27*, p.146) A heptagon is dissected into convex pentagons and hexagons so that each of its vertices belongs to at least two smaller polygons. Prove that the number of polygons in the tesselation is no less than 13.

(11) (Problem 34*, p.149) Given a tree T and two vertices v and w of T, the *distance* $d(v,w)$ *bewteen* v *and* w is the length of the (single) simple path connecting v and w. The *remoteness* $r(v)$ of a vertex v is the sum of the distances between v and all vertices in T: $r(v) = \sum_{w \in T} d(v,w)$. Prove that if a tree has two vertices whose remotenesses differ by 1, then the tree has odd number of vertices.

(12) (Problem 39*, p.149) Each of the edges of a complete graph with 9 vertices is colored either blue or red. Prove that either there exist four vertices with all the edges connecting them blue, or three vertices with all edges connecting them red.

(13) (Problem 40*, p.149) Each of the edges of a complete graph with 10 vertices is colored either black or white. Prove that there are four vertices such that all the edges connecting them are of the same color.

(14) (Problem 53*, p.151) In Orientalia all the roads are one-way roads, and you can reach each town from any other by driving along no more than two roads. One of the roads is closed for repair, but it is still possible to drive from each town to any other. Prove that now this can be done by driving along at most three roads.